Spontaneous scale symmetry breaking in 2+1-dimensional QED at both zero and finite temperature

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Abstract. A complete analysis of dynamical scale symmetry breaking in $2 + 1$-dimensional QED at both zero and finite temperature is presented by looking at solutions to the Schwinger-Dyson equation. In different kinetic energy regimes we use various numerical and analytic techniques (including an expansion in large flavour number). It is confirmed that, contrary to the case of 3+1 dimensions, there is no dynamical scale symmetry breaking at zero temperature, despite the fact that chiral symmetry breaking can occur dynamically. At finite temperature, such breaking of scale symmetry may take place.

1. Introduction

Scale symmetry cannot be an exact symmetry in elementary particle physics, since it would will enforce all the observed particles to be massless or to have a continuous mass spectrum [1]. This explicitly contradicts experimental observation. Therefore, scale invariance must be broken.

In general, there are two kinds of scale symmetry breaking mechanisms. One is explicit breaking which occurs when dimensional parameters are present in the classical action. The other is anomalous scale symmetry breaking which happens at quantum level due to the necessity of implementing renormalization and the consequent occurrence of dimensional transmutation: a new momentum scale automatically arises. In fact, this new scale parameter should be regarded as one of the most important elements underlying a quantum field theory, since in a quantum field theory it is the first derivative of the interaction coupling with respect to the scale parameter, the beta function, rather the coupling itself, that can be explicitly determined. The arising of the scale parameter makes the interaction coupling change with the kinetic energy. Thus in different kinetic energy regions, distinct physical phenomena can be present even though they are dominated by the same theory. One typical example is four-dimensional Quantum Chromodynamics (QCD), where at the high energy relative to the QCD scale parameter, the quarks behave almost as free particles. This asymptotic freedom property of quarks leads to the deep inelastic scattering

cross sections exhibiting a scaling behaviour. At the low-energy level the quarks are confined and chiral symmetry breaking occurs.

However, in some cases spontaneous scale symmetry breaking is also possible. In a quantum field theory with no scalar field such as QCD and spinor electrodynamics etc, spontaneous scale symmetry breaking usually takes place in the strong coupling region near a non-trivial fixed point of the beta function, and occurs in conjunction with spontaneous chiral symmetry breaking [2]. It is well known that the dynamical breaking of chiral symmetry is characterized by the fermion condensation $\langle\bar{\psi}\psi\rangle$, and that its occurrence is determined by the composite operator effective potential generated by quantum corrections, a function of $\langle\bar{\psi}(x)\psi(y)\rangle$ [3]. At the fixed point, the beta function vanishes and the theory will become scale invariant if there are no dimensional parameters present in the classical theory [4]. In particular, the running of the couplings freezes at the fixed point and the anomalous scale symmetry breaking ceases to be a dominant effect. According to the arguments given in [2], if the dynamical chiral symmetry breaking occurs when the anomalous breaking of scale symmetry is not dominant, then chiral symmetry breaking may imply the spontaneous breaking of scale symmetry. It should be emphasized that the dynamical breaking of chiral symmetry does not inevitably result in the spontaneous breaking of scale symmetry, since the instability of the composite operator effective potential under chiral symmetry does not necessarily lead to vacuum degeneracy with respect to scale symmetry. In this case, the dynamical breaking of chiral symmetry only results in the anomalous breaking of scale symmetry. A method to identify spontaneous scale symmetry breaking is to ob-

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serve whether there exists a hierarchy between the scale parameter, which governs the running of the coupling constant and characterizes the anomalous breaking of scale symmetry, and the dynamically generated fermion mass. If there exists such a hierarchy, chiral symmetry breaking can induce spontaneous scale symmetry breaking. Otherwise, the dynamically generated fermion mass only leads to anomalous scale symmetry breaking. The intuitive reason for this is that the scale parameter characterizes the dimensional transmutation and the consequent anomalous scale symmetry breaking. Thus the magnitude of the dynamically generated fermion mass associated with the anomalous scale breaking should be of the same order as the scale parameter, while if the dynamically generated fermion mass has a hierarchy with the scale parameter, it should not attach to the anomalous breaking of scale symmetry and must result from the spontaneous breaking of scale symmetry. A more rigorous but less practical way to judge the spontaneous breaking of scale symmetry is to check the fermion scattering amplitude to see whether there exists a massless scalar particle called a dilaton, since according to Goldstone's theorem, there must arise a massless Goldstone particle with the same properties as the generator of the scale transformation as a consequence of the spontaneous breaking of scale symmetry. One typical example is the scale symmetry breaking connected with the chiral symmetry breaking of four dimensional QCD [2]. Lattice simulation indicates that the scale of chiral symmetry breaking for fermions in higher dimensional representations of the gauge group is much higher than the confinement scale [5]. This fact was further confirmed by checking the effective potential for the composite operator $\langle \bar{\psi}\psi \rangle$ together with the solution of the renormalization group equation [6]. Thus there is spontaneous breaking of scale symmetry associated with fermion condensation.

Some time ago it was found that three-dimensional massless quantum electrodynamics (QED) can exhibit dynamically induced spontaneous chiral symmetry breaking [8,9]. Compared with four-dimensional quantum field theories, three-dimensional QED has several special features. First, it has an intrinsic dimensional parameter, the gauge coupling, that plays the role of the scale parameter in four-dimensional QCD [8]; Second, the beta function of three-dimensional quantum electrodynamics vanishes and hence the coupling constant stays fixed along the whole trajectory of renormalization group flow. Thus there will arise no dimensional transmutation and consequently anomalous scale symmetry breaking does not happen perturbatively. These facts seems to suggest that any spontaneous chiral symmetry breaking would induce spontaneous scale symmetry breaking. However, it was argued implicitly in [8] that there is no spontaneous breaking of scale symmetry at all in three-dimensional QED, despite the occurrence of dynamical chiral symmetry breaking. This situation has caused considerable confusion. In this paper we try to clarify this point, especially at finite temperature. We argue that the introduction of temperature, which carries a mass dimension, drastically changes the zero tem-

perature features of three-dimensional massless QED. In particular, all of the analysis of spontaneous scale symmetry breaking at zero temperature is not valid at finite temperature. Consequently, contrary to the zero temperature case, spontaneously scale symmetry breaking may occur at finite temperature.

2. Scale symmetry of 2+1-dimensional QED

As the first step, it is necessary to clarify the definition of classical scale symmetry. The classical action of massless $2 + 1$-dimensional QED with N_f flavours in the covariant Lorentz gauge is [8]

$$S = \int d^3x \mathcal{L} \tag{1}$$
$$= \int d^3x \left[\sum_{i=1}^{N_f} \bar{\psi}_i \left(i\slashed{\partial} - e\slashed{A} \right) \psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right].$$

The coupling constant has the dimension $(\text{mass})^{1/2}$ and thus the theory is actually superrenormalizable. Further, due to the vanishing beta function, the coupling constant will remain frozen at quantum level. The existence of such a coupling implies that the theory has an explicit scale symmetry breaking. This can be shown as following. Under scale transformation, $x' = e^{-\epsilon}x$, the field transforms according to [1]

$$\phi'(x) = T(\epsilon)\phi(e^\epsilon x); \quad T(\epsilon) = e^{\epsilon d_\phi}$$

where ϵ is the scale transformation parameter and d_ϕ is the scale dimension of the field ϕ. Thus the scale transformations for every fields are

$$\psi'(x) = e^\epsilon \psi(e^\epsilon x), \quad A'_\mu(x) = e^{1/2\epsilon} A_\mu(e^\epsilon x) \tag{2}$$

For an infinitesimal transformation we have

$$\delta \psi_i = \epsilon (1 + x^\mu \partial_\mu)\psi_i,$$
$$\delta (\partial_\mu \psi_i) = \epsilon (2 + x^\nu \partial_\nu)\partial_\mu \psi_i,$$
$$\delta A_\mu = \epsilon \left(\frac{1}{2} + x^\alpha \partial_\alpha \right) A_\mu,$$
$$\delta (\partial_\nu A_\mu) = \epsilon \left(\frac{3}{2} + x^\alpha \partial_\alpha \right) \partial_\nu A_\mu, \tag{3}$$

and consequently

$$\delta S = \epsilon \int d^3x \left[\partial^\mu (x_\mu \mathcal{L}) + \frac{1}{2} e \sum_{i=1}^{N_f} \bar{\psi}_i \slashed{A} \psi_i \right]. \tag{4}$$

The second term on the right hand side of (4) is an explicit violation of scale symmetry. It can be shown, however, that the theory has an approximate scale invariance at scales for which the intrinsic energy scale $e^2 N_f$ can be ignored [3]. The breaking of scale symmetry in (4) is thus classical, and in the following we will examine whether or

not quantum corrections lead to a *dynamical* violation of the scale symmetry.

The conserved quantity corresponding to above special scale symmetry can be defined in the standard way. The general variation of the classical action is

$$\delta S[\phi] = \int d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right.$$
$$\left. + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \right]. \tag{5}$$

For the scale transformation listed in (3), with the fields $\phi = (\bar{\psi}, \psi, A)$ satisfying the classical equations of motion, (3) and (5) yield

$$\int d^3x \, \partial^\mu \left[d_\phi \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \phi + x_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi - g_{\mu\nu}\mathcal{L} \right) \right]$$
$$- \frac{1}{2} e \sum_{i=1}^{N_f} \bar{\psi}_i A \psi_i = 0. \tag{6}$$

Defining the canonical energy-momentum tensor and the dilatation current in the same way as for the four dimensional scale invariant theory,

$$\theta_{\mu\nu}^{(\mathrm{can})} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi - g_{\mu\nu}\mathcal{L},$$
$$d_\mu = d_\phi \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \phi + x^\nu \theta_{\mu\nu}^{(\mathrm{can})}, \tag{7}$$

we can write (6) in the following form,

$$\int d^3x \partial_\mu d^\mu = \int d^3x \frac{1}{2} e \sum_{i=1}^{N_f} \bar{\psi}_i A \psi_i. \tag{8}$$

An explicit calculation gives

$$d_\mu = -\frac{1}{2} F_{\mu\nu} A^\nu - \frac{1}{2\xi} A_\mu \partial_\nu A^\nu + x^\nu \theta_{\mu\nu}^{(\mathrm{can})}, \tag{9}$$

where

$$\theta_{\mu\nu}^{(\mathrm{can})} = \frac{i}{2} \sum_{i=1}^{N_f} \left(\bar{\psi}_i \gamma_\mu \partial_\nu \psi_i - (\partial_\nu \bar{\psi}_i) \gamma_\mu \psi_i \right) - F_{\mu\rho} \partial_\nu A^\rho$$
$$- \frac{1}{\xi} \partial_\nu A_\mu \partial_\rho A^\rho - \delta_{\mu\nu} \left[\frac{i}{2} \sum_{i=1}^{N_f} \left(\bar{\psi}_i \gamma_\rho \partial^\rho \psi_i \right. \right.$$
$$\left. - (\partial^\rho \bar{\psi}_i) \gamma_\rho \psi_i \right) - e \sum_{i=1}^{N_f} \bar{\psi}_i A \psi_i$$
$$\left. - \frac{1}{4} F_{\lambda\rho} F^{\lambda\rho} - \frac{1}{2\xi} (\partial_\rho A^\rho)^2 \right]. \tag{10}$$

Using the classical equations of motion,

$$(i\slashed{\partial} - e\slashed{A})\psi_i = 0, \quad i(\partial^\mu \bar{\psi}_i)\gamma_\mu + e\bar{\psi}_i \slashed{A} = 0,$$
$$\partial^\nu F_{\nu\mu} + \frac{1}{\xi} \partial_\mu (\partial_\alpha A^\alpha) - \sum_{i=1}^{N_f} e\bar{\psi}_i \gamma_\mu \psi_i = 0, \tag{11}$$

we can easily verify

$$\partial^\mu \theta_{\mu\nu}^{(\mathrm{can})} = 0, \quad \theta_\mu^{(\mathrm{can})\mu} = -\mathcal{L} + \sum_{i=1}^{N_f} e\bar{\psi}_i A \psi_i, \tag{12}$$

$$\partial_\mu d^\mu = \theta_\mu^{(\mathrm{can})\mu} + \mathcal{L} - \frac{1}{2} \sum_{i=1}^{N_f} e\bar{\psi}_i A \psi_i$$
$$= \frac{1}{2} \sum_{i=1}^{N_f} e\bar{\psi}_i A \psi_i. \tag{13}$$

When discussing anomalous scale symmetry breaking it is convenient to define an "improved" energy-momentum tensor $\theta_{\mu\nu}$ [10] so that

$$d_\mu = x^\nu \theta_{\mu\nu} - \partial^\nu K_{\mu\nu} \tag{14}$$

with $\theta_{\mu\nu} = \theta_{\nu\mu}$ and $K_{\mu\nu} = -K_{\nu\mu}$. It is easy to show that

$$\partial^\mu \theta_{\mu\nu} = \partial^\mu \theta_{\mu\nu}^{(\mathrm{can})} = 0, \quad \theta_\mu^\mu = \partial_\mu d^\mu = \frac{1}{2} \sum_{i=1}^{N_f} e\bar{\psi}_i A \psi_i. \tag{15}$$

The trace of the energy-momentum tensor will stay the same at the quantum level since the beta function vanishes and no trace anomaly or anomalous scale symmetry breaking arises [4]. The explicit relation between the canonical and the improved energy momentum tensors is not so straightforward as in the case of four-dimensional scalar field theory [10]. It will probably involve a nonlocal form of the fields [11].

The conserved charge corresponding to the scale symmetry is $D = \int d^3x d_0$ and the standard definition of spontaneous scale symmetry breaking is given by $\widehat{D}|0\rangle = 0$, where $\widehat{D}$ denotes the corresponding quantum operator for the dilatation generator. For the dynamical spontaneous breaking of scale symmetry, we should calculate the quantum effective potential composed of the expectation value of the composite operator $\bar{\psi}(x)\psi(y)$ [3] and observe whether the fermionic mass coming from the instability of this effective potential under chiral symmetry has a hierarchy with the scale parameter [2].

The above discussion has clearly shown the difference between scale symmetry and chiral symmetry. Scale transformation invariance is a space-time symmetry, and is much more stringent than chiral symmetry in protecting the theory from receiving quantum corrections: quantum correction can explicitly break scale symmetry much more easily than chiral symmetry. This is the reason that the spontaneous breaking of chiral symmetry does not necessarily lead to the spontaneous breaking of scale symmetry, despite the fact that their common feature is the dynamical generation of a fermion mass term. Only when a further dynamical condition is satisfied (that the dynamical fermionic mass has a hierarchy with the scale parameter) does the dynamical breaking of chiral symmetry imply the spontaneous breaking of scale symmetry.

3. Scale symmetry breaking at zero temperature

To explore the dynamical breaking of chiral symmetry, we need to solve the Schwinger-Dyson equation (SDE) for the fermion self-energy

$$\Sigma(p) = -\not{p}A(p) + B(p), \quad p\equiv|p|. \qquad (16)$$

The SDE for the fermion self-energy can be obtained from extremizing the Cornwall-Jackiw-Tomboulis potential with respect to $B(p)$. Thus, any nontrivial solution to the SDE indicates the spontaneous breaking of chiral symmetry. However, it is notorious that the Schwinger-Dyson equations are a set of closed instantaneous integral equations which are impossible to solve completely. Some appropriate approximation must be utilized. The simplest choice is the rainbow (ladder) approximation, which is consistent with the leading order of the large flavour number N_f expansion [8]. The Ward identity between the fermion self-energy and the fermion-photon vertex requires that $A(p) = 0$ under this approximation. In the rainbow approximation the gap equation in Landau gauge ($\xi = 0$) reduces to [8,9],

$$B(p) = \frac{e^2}{2\pi^2 p} \int_0^\infty dq \frac{qB(q)}{q^2 + B^2(q)} \ln \frac{p+q+N_f e^2/8}{|p-q|+N_f e^2/8}. \quad (17)$$

For $p\ll N_f e^2$ or $p\gg N_f e^2$ the above integral equation can be converted into a second order nonlinear differential equation [8]

$$\frac{d}{dp}\left[\frac{dB(p)}{dp}\frac{p^2(p+N_f e^2/8)^2}{2p+N_f e^2/8}\right] = -\frac{N_f e^2}{\pi^2 N_f}\frac{p^2 B(p)}{p^2+B^2(p)}. \quad (18)$$

The fermion condensate is given by

$$\langle\bar{\psi}(p)\psi(-p)\rangle \sim \text{Tr}[-\not{p}+B(p)]^{-1} = \frac{4B(p)}{p^2+B^2(p)}. \quad (19)$$

In the region $B(p)\ll p\ll N_f e^2$ (18) can be linearized and an analytical solution can be found. When $N_f < 32/\pi^2$, the solution has the form

$$B(p) = \frac{(N_f e^2)^{3/2}}{\sqrt{p}}$$
$$\times \sin\left[\frac{1}{2}\sqrt{\frac{32}{N_f\pi^2}-1}\ln\left(\frac{p}{B(0)}\right)+\delta\right], \quad (20)$$

where δ is a phase factor which may depend on the parameter N_f. The boundary conditions

$$0\leq B(0) < \infty; \quad \lim_{p\to N_f e^2}\left[p\frac{dB(p)}{dp}+B(p)\right] = 0 \quad (21)$$

together with (20) determine that the dynamically generated mass defined at $p = 0$ is [8]

$$B(0)\sim N_f e^2 \exp\left[-\frac{2(n\pi-\delta)}{\sqrt{32/(\pi^2 N_f)-1}}\right]. \qquad (22)$$

This solution demonstrates the existence of a critical flavour number $N_f^c = 32/\pi^2$ for the restoration of chiral symmetry.

Despite some dispute about the existence of N_f^c [12], a striking feature of (22) is that the dynamically generated mass is of the order of $N_f e^2$. In fact, this result is not unexpected since no dimensional transmutation occurs in QED_3 and thus there is no dynamically generated momentum scale. The dynamical mass can only be associated with the coupling constant, since the coupling constant is the only scale parameter of the theory.

In the region $p\gg N_f e^2$ (18) cannot be linearized, but the asymptotic form of the solution has been obtained [8],

$$B(p)\sim\frac{(N_f e^2)^3}{p^2}\left[1-\left(\frac{1}{8}+\frac{2}{3N_f\pi^2}\right)\frac{N_f e^2}{p}+\cdots\right], (23)$$

and the solution for the truncated lower-integral equation of (17) is [8]

$$B(p)\sim\frac{(N_f e^2)^3}{p(p+N_f e^2/8)}\left[\frac{p}{p+N_f e^2/8}\right]^{8/(\pi^2 N_f)}. \quad (24)$$

The solution (24) is only valid for $p\gg N_f e^2$ and becomes unreliable when $p\sim N_f e^2$. Equations (23) and (24) imply that $B(p)\sim(N_f e^2)^3/p^2$. This result represents a dynamically generated mass function and thus leads to chiral symmetry breaking. Note that the solution (24) has nothing to do with the $1/N_f$ expansion since this expansion does not play any role in this kinetic energy region.

When $p\sim N_f e^2$ it is difficult to get an analytical information about the solution to the SDE (17) and the only way to proceed is to use a numerical simulation. The numerical solution given in [8] shows that when p goes past $N_f e^2$ moving in the direction of increasing p, $B(p)$ sharply falls to zero. In combination with the behaviour of the solutions to the SDE in the cases $p\ll N_f e^2$ and $p\gg N_f e^2$, one concludes that in the region $p\sim N_f e^2$, $B(p)$ changes smoothly from its slowly falling form at $p\ll N_f e^2$ to a sharply falling form at $p\gg N_f e^2$ [8].

Based on these results, we can observe whether or not a dynamical violation of scale symmetry occurs, keeping in mind the discussion after (4) concerning the this symmetry at the classical level. In the momentum range $p \ll N_f e^2$ (22) indicates that the dynamical fermion mass is proportional to $N_f e^2$ and thus that there is no hierarchy between the energy scale $N_f e^2$ and the dynamical mass $B(0)$. Therefore, it is not possible for a dynamical violation of scale symmetry to take place. Note that if we approach the critical flavour number, the dynamical mass approaches zero and chiral symmetry breaking is lost, which means that there is still no possibility of dynamically broken scale symmetry. In the momentum range $p \gg N_f e^2$ we again have spontaneous chiral symmetry breaking since the dynamical mass function $B(p)$ behaves like $1/p^2$. However there exists no hierarchy between the dynamical mass and the scale $N_f e^2$ in all cases and hence no dynamical violation of scale symmetry breaking occurs.

4. Scale symmetry breaking at finite temperature

Despite the fact that there is no spontaneous breaking of scale symmetry at zero temperature, it is possible that it could take a place at finite temperature. An intuitive reason for this is as follows: temperature is a parameter with mass dimension and thus the dynamically generated mass may depend not only on the coupling constant, but also on this new dimensional parameter. This new dimensional parameter makes it possible that there will be a hierarchy between the dynamical mass and the momentum scale $N_f e^2$ provided by the coupling constant. Moreover, the coupling constant at the quantum level may be temperature dependent, and hence contrary to the zero temperature case, there will arise a non-vanishing beta function which depends on the temperature. If this beta function has a non-trivial fixed point, then a similar observation as in the case of four-dimensional QED at zero temperature could reveal the occurrence of the spontaneous breaking of scale symmetry.

As in the zero-temperature case, we make use of the Schwinger-Dyson equation to observe the dynamical mass function. The Schwinger-Dyson equation for the fermion self-energy at finite temperature $\Sigma(p_0, |\mathbf{p}|)$, in Landau gauge reads [12]

$$\Sigma(p_0, |\mathbf{p}|) = -p_0 \gamma_0 A(p_0, |\mathbf{p}|) - \mathbf{p} \cdot \gamma A(p_0, |\mathbf{p}|) + B(p_0, |\mathbf{p}|)$$
$$= -\frac{e^2}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \gamma_\mu S(q_0, |\mathbf{q}|) \Gamma_\nu D^{\mu\nu}$$
$$\times (p_0 - q_0, |\mathbf{p} - \mathbf{q}|, \beta), \qquad (25)$$

where $\beta = 1/(k_B T)$, $p_0 = (2m+1)\pi/\beta$ and $q_0 = (2n+1)\pi/\beta$. To leading order in the $1/N_f$ expansion, the ladder approximation is appropriate: Γ_μ can be replaced by the bare vertex $e\gamma_\mu$, and the propagator $D_{\mu\nu}(p_0, \mathbf{p}, \beta)$ is approximately evaluated in the chain approximation of fermionic loop [12]. The Ward identity further requires that $A(p_0, |\mathbf{p}|) = 0$. The dynamical mass function is obtained by taking the trace of (25)

$$B(p_0, |\mathbf{p}|, \beta) = \frac{e^2}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2\mathbf{q}}{(2\pi)^2} D(p_0 - q_0, |\mathbf{p} - \mathbf{q}|, \beta)$$
$$\times \frac{B(q_0, |\mathbf{q}|, \beta)}{q^2 + B^2(q_0, |\mathbf{q}|, \beta)} \qquad (26)$$

where $D(p_0, |\mathbf{p}|, \beta) \equiv \text{Tr}[\gamma_\mu D_{\mu\nu}(p_0, |\mathbf{p}|, \beta)\gamma_\nu]/8$ [13]. The closed integral equation (26) for $B(p_0, |\mathbf{p}|, \beta)$ has been numerically solved in an instantaneous exchange approximation in which the p_0 dependence of the vacuum polarization tensor has been ignored. As a consequence, $B(p_0, |\mathbf{p}|, \beta)$ also becomes frequency independent. In the approximation of considering only the $\mu = \nu = 0$ component, the propagator takes the form [12],

$$D_{\mu\nu}(\mathbf{p} - \mathbf{q}, \beta) = \frac{\delta_{\mu 0}\delta_{\nu 0}}{|\mathbf{p} - \mathbf{q}|^2 + \Pi(|\mathbf{p} - \mathbf{q}|, \beta)}, \qquad (27)$$

and

$$\Pi(|\mathbf{q}|, \beta) = \frac{2N_f e^2}{\pi\beta} \qquad (28)$$
$$\times \int_0^1 dx \ln\left\{2\cosh\left[\beta/2|\mathbf{q}|\sqrt{x(1-x)}\right]\right\}.$$

After the summation over n is performed, (26) becomes [13]

$$B(|\mathbf{p}|, \beta) = \frac{e^2}{8\pi^2} \int d^2\mathbf{q} \frac{B(|\mathbf{q}|, \beta)}{(\mathbf{p} - \mathbf{q})^2 + \Pi(|\mathbf{p} - \mathbf{q}|, \beta)}$$
$$\times \frac{\tanh\left[\beta/2\sqrt{\mathbf{q}^2 + B^2(|\mathbf{q}|, \beta)}\right]}{\sqrt{\mathbf{q}^2 + B^2(|\mathbf{q}|, \beta)]}}. \qquad (29)$$

The numerical solutions of (29) have been explicitly obtained in the kinetic energy regions $|\mathbf{p}| < k_B T < N_f e^2$ and $k_B T < |\mathbf{p}| < N_f e^2$ [13]. The analysis given below will show that these are the only two regions in which dynamical chiral symmetry breaking takes place, thus they are the only regions that need to be considered. The numerical solution shows that there exists both a critical temperature and a critical flavour number, above which the dynamical mass vanishes and chiral symmetry breaking is restored. The existence of a critical temperature implies that we do not expect chiral or scale symmetry breaking at high temperature, and thus that it is sufficient to consider the range $k_B T < k_B T_c < N_f e^2$.

In the kinetic energy region $|\mathbf{p}| < k_B T < N_f e^2$, the dynamical mass defined at $|\mathbf{p}| = 0$ is a function of $N_f e^2$ and $k_B T$. Due to some special features of the numerical calculation, the numerical solution can only describe the dependence of the mass on one of the two independent parameters (T and N_f) at a time. When we hold the temperature T fixed, the numerical solution implies that the dynamical mass takes the following form [13]

$$B(0, \beta, e^2 N_f) \propto N_f e^2 \exp\left[-\frac{C(T)}{\sqrt{N_c(T)/N_f - 1}}\right], \quad (30)$$

where $C(T)$ is a certain temperature dependent function and $N_c(T)$ is the critical flavour number (since $B(0, \beta, e^2 N_f) \to 0$ as $N_f \to N_c$). Equation (30) seems to suggest that no spontaneous breaking of scale symmetry is induced at any temperature since no matter how the temperature varies, the dynamical mass is always of order $N_f e^2$. However, when we fix N_f and look at the numerical solution for the dynamical mass as a function of T we have [13],

$$B(0, \beta, e^2 N_f) \propto (N_f e^2)^{1-x(N_f)} [k_B (T_c - T)]^{x(N_f)};$$
$$k_B T \sim 10^{-3} N_f e^2. \qquad (31)$$

It has been shown that when $1 < N_f < 2$, the exponent x has the value $0.4 < x(N_f) < 0.6$ [13]. Equation (31) not only explicitly shows the existence of the critical temperature T_c, but also reveals that when $T \to T_c$ the dynamical mass $B(0, \beta, N_f e^2)$ is very small in comparison

with the intrinsic energy scale $N_f e^2$. Thus a gap between the dynamical mass and the energy scale $N_f e^2$ appears. This observation implies that spontaneous scale symmetry breaking occurs in the region $|\mathbf{p}| < k_B T < N_f e^2$. Note also that, as stated above, the solution (31) indicates the existence of a critical temperature T_c and thus that at the high temperature $T > T_c$ there will be no spontaneous breaking of chiral symmetry or of scale symmetry.

When $k_B T < |\mathbf{p}| \leq N_f e^2$, the dynamical mass cannot be defined at $|\mathbf{p}| = 0$. For certain fixed temperatures, the numerical solution shows that the ratio between the dynamical mass function and $N_f e^2$ decreases to zero when $|\mathbf{p}| \sim N_f e^2$ [13] which indicates that there is no chiral symmetry breaking. When $k_B T < |\mathbf{p}| \ll N_f e^2$, if the flavour number is big enough, the numerical solution indicates that $B(|\mathbf{p}|, \beta) \sim N_f e^2$. There is again no dynamical violation of scale symmetry. However, when the flavour number is small, the numerical solution shows that there exists a big gap between the dynamical mass and the scale $N_f e^2$, which suggests a dynamical violation of scale symmetry.

There is no information about the dynamical mass in the the kinetic energy region $|\mathbf{p}| > N_f e^2 > k_B T$. However, if the dynamical mass function is a continuous and monotonically decreasing function of $|\mathbf{p}|$, then the numerical solution at $|\mathbf{p}| \sim N_f e^2$ implies that the dynamical mass should approach zero in this kinetic energy region. Therefore in this region chiral symmetry should be restored, and there should be no dynamical violation of scale symmetry.

The Schwinger-Dyson equation (26) has been solved numerically beyond the instantaneous exchange approximation, but the main features of the solutions remain unchanged [14]. Therefore, the above analysis on the spontaneous breaking of scale symmetry is likely to be valid beyond the instantaneous exchange approximation.

5. Summary and discussion

Keeping in mind the discussion after (4) concerning the nature of scale symmetry at the classical level, we have found that the dynamical breaking of scale symmetry is a very delicate non-perturbative phenomenon. Its occurrence is not easy to identify since in most situations anomalous scale symmetry breaking prevails. In this paper we have given a detailed analysis of spontaneous scale symmetry breaking in 2+1-dimensional QED based on solutions to the Schwinger-Dyson equations for the fermion self-energy at both zero and finite temperature. In the case of zero temperature we show explicitly that scale symmetry breaking cannot be dynamically induced despite the fact that chiral symmetry breaking occurs. The main reasons for this are the super-renormalizability of the theory and the perturbative ultraviolet finiteness of $2 + 1$ dimensional QED. These two facts eliminate the possibility for dimensional transmutation to occur and thus the only available scale parameter is the coupling constant. Consequently, the dynamical mass must be proportional to the square of the coupling constant. The explicit solutions of the SDE show that spontaneous breaking of scale

symmetry, in the sense of (4) and the ensuing discussion, does not occur.

The finite temperature case contains new physics. The dynamical spontaneous breaking of chiral symmetry only occurs in the kinetic energy region $|\mathbf{p}| < N_f e^2$, and only in the case $k_B T < N_f e^2$ with $k_B T < k_B T_c$ and $N_f e^2 < N_c e^2$ (N_c and T_c being the critical flavour number and critical temperature to restore chiral symmetry). When $|\mathbf{p}| < k_B T < N_f e^2$, the numerical solution to the Schwinger-Dyson equation reveals that scale symmetry breaking may be induced dynamically, since there arises a hierarchy between the dynamical mass and the energy scale $N_f e^2$. When $k_B T < |\mathbf{p}| < N_f e^2$, in the case of small flavour number, there is also an indication that spontaneous scale symmetry breaking may take place.

It should be emphasized, however, that the above conclusions about the phase structures of scale symmetry are based on numerical solutions which are obtained by making some specific choices for the values of parameters and techniques used to solve the Schwinger–Dyson equation (16). Other, more elaborate ansätzes, such as ones involving more complicated vertex functions and subsequently non–trivial fermionic wave–function renormalizations, can also be used, and provide a valuable check on the consistency and completeness of these solutions [14]. As such, these results should only be viewed as qualitative. A rigorous method to judge the spontaneous breaking of scale symmetry is to calculate the fermion scattering amplitude and observe whether it possesses a pole indicating the existence of the dilaton.

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